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On a problem of R. Brauer for quotients of Dedekind zeta-functions

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INTRODUCTION

In this paper all groups will be finite and all fields will be extensions of finite degree over the rational field $\mathbb{Q}$. The character 1_G stands for the *principal character* of the group G. If θ is a character of a subgroup of G, then θ^G will mean the character of G which is *induced* by θ. If ψ is a character of G, then ψ_H will be the character of H obtained by *restriction* of ψ to the subgroup H of G. Let $\zeta_T(s)$ denote the *Dedekind zeta-function* of the field T. Then $\zeta_{K,L}(s)$ will stand for

$$\frac{\zeta_{KL}(s)\cdot\zeta_{K\cap L}(s)}{\zeta_K(s)\cdot\zeta_L(s)},$$

where KL is the *compositum* of the fields K and L.

Group-theoretical notation is standard or self-explanatory; see [3] and [4]. For properties on Artin L-functions the reader is referred to J. Martinet's contribution in ([2], pages 1-20).

Let D/F be a galois extension and let E be an intermediate field between D and F. Put $G=\mathrm{Gal}(D/F)$ and $H=\mathrm{Gal}(D/E)$. The Dedekind zeta-function $\zeta_E(s)$ of E is the same as the Artin L-function $L(s,1_H,D/E)$ attached to the principal character 1_H of H. We have $\zeta_E(s)=L(s,1_H,D/E)=L(s,(1_H)^G,D/F)$. Therefore, if K is another intermediate field between D and F with $T:=\mathrm{Gal}(D/K)$, it follows from known properties of Artin L-functions that

$$\zeta_{E,K}(s) = \frac{L(s, 1_{H\cap T}, D/EK) \cdot L(s, 1_{\langle H,T\rangle}, D/(E\cap K))}{L(s, 1_H, D/E) \cdot L(s, 1_T, D/K)}$$
$$= L(s, (1_{H\cap T})^{\langle H,T\rangle} + 1_{\langle H,T\rangle} - (1_H)^{\langle H,T\rangle} - (1_T)^{\langle H,T\rangle}, D/(E\cap K));$$

note that EK is the fixed field of $H \cap T$ and that $E \cap K$ is the fixed field of $\langle H, T\rangle$, the subgroup of G generated by H and T.

Observe that

$$\chi := (1_{H\cap T})^{\langle H,T\rangle} + 1_{\langle H,T\rangle} - (1_H)^{\langle H,T\rangle} - (1_T)^{\langle H,T\rangle}$$

is a *character* of $\langle H, T\rangle$. This observation is due to R. Brauer; for a proof see ([4], Theorem (5.19)). It is also easily seen that 1_G is not an irreducible constituent of χ.

Now, if it happens that χ is a *non-trivial, non-negative rational sum of monomial characters of* $\langle H, T\rangle$ (i.e. $\chi = a_1\vartheta_1 + \cdots + a_t\vartheta_t$, $t \geq 1$, $a_i \in \mathbb{Q}_{\geq 0}$ for $i = 1, \ldots, t$, not all $a_i = 0$, and ϑ_i a character of $\langle H, T\rangle$ such that $\vartheta_i = (\lambda_i)^{\langle H,T\rangle}$ for some 1-dimensional character λ_i of a suitable subgroup H_i of $\langle H, T\rangle$), then it follows from the holomorphy of Artin L-functions attached to non-principal monomial characters that $\zeta_{E,K}$ is an *entire function*. In this way it was proved by R. Brauer ([1], Theorem) that $\zeta_{E,K}(s)$ is entire whenever H and T are proper *normal* subgroups of $\langle H, T\rangle$.

We say that *Brauer's problem has an affirmative answer* (for the fields A and B), if it turns out that $\zeta_{A,B}(s)$ *is an entire function*. The second author of this paper obtained a list of fields for which Brauer's problem can be answered in the affirmative. See the references [5.a]–[5.g].

In analogy to the proof of ([1], Theorem) there was some hope that an affirmative answer to Brauer's problem could be deduced from character theory of groups. It turns out however, that this hope is in vain, even for the solvable group case. It is demonstrated in Example 7 that for the solvable group $SL(2,3)$, being generated by two subgroups S and T each of order 3, it does not hold that $(1_{S\cap T})^{SL(2,3)} + 1_{SL(2,3)} - (1_S)^{SL(2,3)} - (1_T)^{SL(2,3)}$ is a non-negative, non-trivial rational sum of monomial characters of $SL(2,3)$. At this point, however, one is invited to read the Remark after Example 7.

We will show that there are cases of a general nature in which an affirmative answer to Brauer's problem can be given in which solvable groups are involved. See Theorems 5 and 6 in §1. For instance, the following is a direct consequence to Theorem 6.

THEOREM 6′. *Suppose K and L are distinct field extensions of F. Suppose also that there are no intermediate fields between K and F and between L and F. Assume there exists a galois extension M/F with $M \supseteq KL$, in such a way that* $\mathrm{Gal}(M/F)$ *is solvable. Then Brauer's problem has an affirmative answer for K and L.*

We treat also the cases $G = A_5$ (a simple group) and the solvable group $GL(2,3)$. What happens then, is also one of the topics in this paper.

The outcome of this paper originated from conversations of K. Sato with R.M. Foote (summer 1990) and of R.W. van der Waall with K. Sato (summer 1991).

The Theorems 3, 5 and 6 are due to R.W. van der Waall; Theorem 4 is due to R.M. Foote, whereas the formulation of its proof has been modified by R.W. van der Waall.

Example 7 has been found by K. Sato by examining the character theory of $SL(2,3)$; R.W. van der Waall streamlined the argumentation.

Example 8 has been deduced by R.W. van der Waall from a long list of pairs of subgroups of A_5, drawed up by K. Sato, treating thereby the corresponding character χ as defined above.

Example 9 has been established by K. Sato after conclusive discussions with R.W. van der Waall.

1. THE THEOREMS AND THEIR PROOFS

We start with some preliminary work that will enable us to prove the Theorems 5 and 6.

DEFINITION. Let χ be a character of G. We say that χ is a monomial character of G if there exists some subgroup H of G and a linear character λ of H such that χ is induced by λ.

LEMMA 1 ([3], Satz V.16.8). *If τ is a class function on G and ψ is a class function on the subgroup R of G, then $\psi^G \cdot \tau = (\psi \cdot \tau_R)^G$.* □

LEMMA 2 ([3], Satz V.16.9). *If H and K are subgroups of G and $G = \bigcup_{i=1}^{k} Ha_iK$ is the decomposition of G in pairwise disjoint double cosets with respect to H and K, and if χ is a character of H, then*

$$(\chi^G)_K = \sum_{i=1}^{k} ((\chi^{a_i})_{a_i^{-1}Ha_i \cap K})^K;$$

here χ^{a_i} is the character of $a_i^{-1}Ha_i$ defined by $\chi^{a_i}(g) = \chi(a_i g a_i^{-1})$ for $g \in a_i^{-1}Ha_i$. □

An important consequence to the Lemmas 1 and 2 is the following theorem. Probably its content was unnoticed sofar.

THEOREM 3. *Let μ be a monomial character of G. Let K be a subgroup of G. Then μ_K is a sum of monomial characters of K. In addition, if ν is a monomial character of G, then $\mu \cdot \nu$ is a sum of monomial characters of G.*

PROOF. The character μ is induced by some linear character λ of a suitable subgroup H of G. Hence it follows from Lemma 2 that μ_K is a sum of monomial characters of K. By the same token $\nu_H = \sum_{i=1}^{r} \alpha_i$, where $r \geq 1$ and each α_i is a monomial character of H. Put $\alpha_i = (\xi_i)^H$ for some linear character

ξ_i of suitable subgroup H_i of H. Then we conclude from Lemma 1 that

$$\mu \cdot \nu = (\lambda \cdot \nu_H)^G = (\lambda \cdot (\sum_{i=1}^{r} \alpha_i))^G$$

$$= (\sum_{i=1}^{r} (\lambda_{H_i} \cdot \xi_i)^H)^G = \sum_{i=1}^{r} (\lambda_{H_i} \cdot \xi_i)^G.$$

So, as $\lambda_{H_i} \cdot \xi_i$ is a linear character of H_i for $i = 1, \ldots, r$, we see that $\mu \cdot \nu$ is a sum of monomial characters of G. $\square$

It is a theorem, due to van der Waall and Uchida, that $\zeta_L(s)/\zeta_K(s)$ is an entire function in case there exists a galois extension $\Omega \supseteq K$ with $\Omega \supseteq L \supseteq K$ such that $\mathrm{Gal}(\Omega/K)$ is *solvable* (See ([2], pages 649–662, for a survey on the history of this topic). It turns out to be a consequence of the following group theoretical theorem. Note that Theorem 4 as it stands, has not been stated in either [7] or [9]. We will need Theorem 4 in the proof of the Theorems 5 and 6.

THEOREM 4. *Let G be a solvable group. H a proper subgroup of G. Then $(1_H)^G - 1_G$ is a sum of monomial characters of G. Moreover, 1_G is not an irreducible constituent of $(1_H)^G - 1_G$.*

PROOF. Let R be a maximal subgroup of G containing H. As any subgroup of G is solvable, we may infer by induction that $((1_H)^R - 1_R)^G = (1_H)^G - (1_R)^G$ is a sum of monomial characters of G. So we are left to prove that $(1_R)^G - 1_G$ is a sum of monomial characters of G.

Now, as G is solvable, there exists an abelian $A \trianglelefteq G$ with $A \neq \{1\}$ such that either $A \leq R$ or else $G = RA$.

Assume $A \leq R$. As G/A is solvable we may assume that $(1_R)^G - 1_G$ is a sum of monomial characters μ_i of G with $\mathrm{Ker}\, \mu_i \geq A$, and we are done.

So let $G = RA$. Then $A \cap R$ is an abelian normal subgroup of G. Hence we may assume that $A \cap R = \{1\}$. In that case $(1_R)^G - 1_G$ is a sum of pairwise distinct irreducible monomial characters of G, as shown in [7] and [9].

Direct calculation shows that indeed $(1_G, (1_H)^G - 1_G) = 0$. $\square$

After these preparatory results we are able to show that Theorem 5 gives an affirmative answer to Brauer's problem in the situation we are after here.

THEOREM 5. *Suppose H and T are proper subgroups of the solvable group G and that $G = HT$ (i.e. each element of G is of the form ht for suitable $h \in H$, $t \in T$). Then $\chi := (1_{H \cap T})^G + 1_G - (1_H)^G - (1_T)^G$ is a sum of monomial characters of G. Furthermore, 1_G is not an irreducible constituent of χ.*

PROOF. Let Lemmas 1 and 2 yield

$$(1_{H \cap T})^G = ((1_{H \cap T})^T)^G$$

$$= (((1_H)^{HT})_T)^G = ((((1_H)^G)_T \cdot 1_T)^G = (1_H)^G \cdot (1_T)^G.$$

Therefore $\chi = ((1_H)^G - 1_G) \cdot ((1_T)^G - 1_G)$. Now $(1_H)^G - 1_G$ is a sum of monomial characters α_i of G (say), and $(1_T)^G - 1_G$ is a sum of monomial characters β_i of G (say), all this by Theorem 4. Hence

$$\chi = \sum_i \sum_j (\alpha_i \cdot \beta_j).$$

Observe that each character of that shape $\alpha_i \cdot \beta_j$ is a sum of monomial characters of G, by Theorem 3. It is clear that 1_G is not an irreducible constituent of χ. This proves the Theorem. $\square$

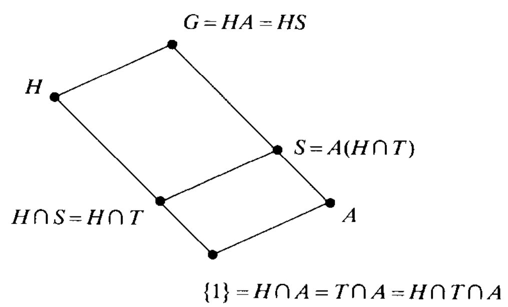
Let $G \neq \{1\}$ be generated by the *maximal* subgroups K and L of G. Then it will be shown in the next theorem that Brauer's problem has an answer in the affirmative for G with respect to those subgroups K and L, whenever G is *solvable*.

THEOREM 6. *Let $G \neq \{1\}$ be a solvable group, generated by maximal subgroups H and T of G. Then $\chi := (1_{H \cap T})^G + 1_G - (1_H)^G - (1_T)^G$ is a sum of monomial characters of G (and 1_G is not an irreducible constituent of χ).*

PROOF. If H is not conjugate to T in G, then $G = HT$ by Ore's theorem ([3], Satz II.3.9). Thus Theorem 5 yields the result.

From now on we will assume that H and T are conjugate in G. There exists an abelian $A \trianglelefteq G$ with $A \neq \{1\}$, as G is solvable. If $A \leq H \cap T$, then we conclude that the theorem holds by induction, for everything has to do with character theory of G/A. Thus we can now assume that $G = AH = AT$. Then note that $A \cap H = A \cap T = A \cap H \cap T$ and that $A \cap H$ is an abelian normal subgroup of G. Hence we are allowed to assume from now on that also $A \cap H = \{1\}$, by the same induction principle as before.

Put $S = A(H \cap T)$; note that $G > S > H \cap T$. Observe that $HS = H(A(H \cap T)) = G$, that $G = HA$, and that $H \cap S = H \cap T$.



We see from the proof of ([9], Theorem) regarding the structure of $((1_H)^G)_A = (1_{\{1\}})^A$, that there exist non-principal monomial irreducible characters $\beta_1, \ldots, \beta_u$ of G with $\beta_i \neq \beta_j$ whenever $i \neq j$, such that $(1_H)^G = 1_G + \beta_1 + \cdots + \beta_u$; note $u \geq 1$. Observe that $((1_H)^G)_S = ((1_H)^{HS})_S = (1_{H \cap S})^S = (1_{H \cap T})^S$; note also that $(1_H)^G = (1_T)^G$ as H and T are conjugate subgroups of G. Therefore

$$\begin{aligned}\chi &= (1_{H\cap T})^G + 1_G - (1_H)^G - (1_T)^G \\ &= (1_S + (\beta_1)_S + \cdots + (\beta_u)_S)^G + 1_G - 2(1_H)^G \\ &= (1_S)^G - 1_G + \sum_{i=1}^{u} (((\beta_i)_S)^G - 2\beta_i).\end{aligned}$$

Now $(1_S)^G - 1_G$ is a sum of monomial characters of G, by Theorem 4. We will show that for any i $((\beta_i)_S)^G - 2\beta_i$ is a sum of monomial characters of G or that $((\beta_i)_S)^G = 2\beta_i$.

There exists a suitable $R \leq G$ admitting a linear character λ with $\lambda^G = \beta_1$. Due to Seitz's theorem ([6], Lemma 1.8), R can be selected such that $R \geq A$, and we will do so. From Lemma 2 it follows that

$$(\beta_1)_S = (\lambda^G)_S = \sum_{a_i} ((\lambda^{a_i})_{a_i^{-1}Ra_i \cap S})^S;$$

here the a_i constitute a full set of representatives for the double cosets in G with respect to R and S; put $a_1 = 1$. The linear character λ^{a_i} of $a_i^{-1}Ra_i$ is related to λ by means of $\lambda^{a_i}(a_i^{-1}ra_i) = \lambda(r)$ for $r \in R$, whence $(\lambda^{a_i})^G = \lambda^G$ for any i. As the regular character of the abelian group A is equal to $((1_H)^G)_A$, we see that $((\beta_1)_A, \mu) \leq 1$ for any $\mu \in \mathrm{Irr}(A)$. So, as any $a_i^{-1}Ra_i \cap S$ contains A, it follows from Frobenius' reciprocity that $((\lambda^{a_i})_{a_i^{-1}Ra_i \cap S})^S \in \mathrm{Irr}(S)$ for any i. We deduce from ([9], Theorem) that $(1_{\{1\}})^A = ((1_{H\cap T})^S)_A$ yields $(1_{H\cap T})^S = 1_S + \alpha_1 + \cdots + \alpha_t$, where each α_i is a monomial irreducible character of S with $\alpha_j \neq \alpha_k$ whenever $j \neq k$; note $t \geq 1$ by $S > H \cap T$. So, as $((1_H)^G)_S = (1_{H\cap T})^S$, we are able to compare the irreducible constituents of $(\beta_i)_S$ with the α_i's.

We now show that $(\beta_i)_S$ is not irreducible, as follows. Since $G = HS$, $(1_H)^G$ and $(1_S)^G$ do not have irreducible constituents in common, except for 1_G and 1_G occurs precisely once in $(1_H)^G$ and in $(1_S)^G$. Hence $((\beta_1)_S, 1_S) = (\beta_1, (1_S)^G) = 0$. Hence $((\beta_1)_S, \alpha_k) = 1$ for at least one $k \in \{1, \ldots, t\}$. Now, just as $G = \langle H, T\rangle$ and $(1_H)^G = (1_T)^G$, it follows from a result of Brauer (see [4], Theorem 5.19) that $(\beta_1, (1_{H\cap T})^G) \geq 2$. Any of the α_i occurs in $(\beta_1)_S$ with multiplicity at most one, as $((\beta_1)_A, \mu) \leq 1$ for $\mu \in \mathrm{Irr}(A)$. Therefore, as $(\beta_1, (1_S)^G) = 0$, we see that there must exist α_l different from α_k such that $((\beta_1)_S, \alpha_l) = 1$. Hence indeed $(\beta_i)_S \notin \mathrm{Irr}(S)$ for any i.

We have seen above that there exists $a \in G$ such that $\alpha_k = ((\lambda^a)_{a^{-1}Ra \cap S})^S$ is an irreducible constituent of $(\beta_1)_S$. Therefore

$$\begin{aligned}(\alpha_k)^G - \beta_1 &= (((\lambda^a)_{a^{-1}Ra\cap S})^S)^G - \lambda^G \\ &= (((\lambda^a)_{a^{-1}Ra\cap S})^{a^{-1}Ra})^G - (\lambda^a)^G \\ &= (\lambda^a \cdot ((1_{a^{-1}Ra\cap S})^{a^{-1}Ra} - 1_{a^{-1}Ra}))^G \\ &= (\lambda^a \cdot (\text{zero character of } a^{-1}Ra \text{ or a sum of monomial} \\ &\qquad \text{characters of } a^{-1}Ra))^G \\ &= (\text{sum of monomial characters of } G) \text{ or } (\text{zero character of } G).\end{aligned}$$

The same result is true for $(\alpha_l)^G - \beta_1$.

Therefore it follows from $((1_H)^G)_S=(1_{H\cap T})^S$, that

$$(\beta_1)_S=\alpha_k+\alpha_l+X,$$

where X is a sum of monomial characters of S or X is the zero character of S. Hence

$$((\beta_1)_S)^G-2\beta_1=((\alpha_k)^G-\beta_1)+((\alpha_l)^G-\beta_1)+M,$$

where M is a sum of monomial characters of G or it is the zero character of G.

Hence indeed $((\beta_i)_S)^G-2\beta_i$ is equal to a sum of monomial characters of G or it is the zero character of G, for any $i\in\{1,\dots,u\}$. The property $(\chi,1_G)=0$ follows from the definition of χ.

The Theorem has been proved. $\square$

For the sake of completeness we mention that the second author of this paper proved (unpublished; summer 1990) that

$$\varrho+1_G-(1_H)^G-(1_T)^G$$

is a sum of monomial characters of G, whenever G is solvable, H and T maximal subgroups of G, $G=HT$; here ϱ means the regular character of G. In the remaining case (i.e. $G=\langle H,T\rangle$, H and T maximal subgroups of G, G solvable, H and T conjugate in G) the analogous result is due to R.M. Foote (also unpublished; summer 1990). Remember here Ore's theorem ([3], Satz II.3.9).

Both situations are incorporated in the Theorem 5 and 6, respectively. Namely, $\varrho=((1_{\{1\}})^{H\cap T}-1_{H\cap T})^G+(1_{H\cap T})^G$, and Theorem 4 yields that if $H\cap T\neq\{1\}$, $((1_{\{1\}})^{H\cap T}-1_{H\cap T})^G$ is a sum of monomial characters of the solvable group G. By the same token it is clear now that $(1_K)^G+1_G-(1_H)^G-(1_T)^G$ is a sum of monomial characters of the solvable group G, whenever $K\leq H\cap T$ and $G=\langle H,T\rangle$.

To close our comments on the Theorems 5 and 6 we mention that we have proved for number fields that the following is true.

THEOREM 5′. *Suppose that K and L are fields for which the degree $[KL:K]$ is the same as the degree $[L:(K\cap L)]$. If there exists a field Ω containing both K and L such that $\Omega/(K\cap L)$ is a galois extension with solvable galois group, then* $\zeta_{K,L}(s)$ is an entire function.

2. THE EXAMPLES

After the promising results around Brauer's problem, there could be some hope that the 'solvable group'-case might give rise to an affirmative answer to Brauer's problem. It turns out, however, that this hope is in vain. So, in order to achieve a positive answer to Brauer's problem for all number fields, there has to be found something else going beyond the concept of the theory on monomial characters.

EXAMPLE 7. Let $G=SL(2,3)$ be the special linear group of 2×2-matrices over the field of three elements. There exists $Q \trianglelefteq G$, Q isomorphic to the quaternion group of order 8, and a cyclic group C of order 3, such that $G = QC$. There are other subgroups of G of order 3. Take one, say F. We have $G = \langle C, F \rangle$. As C and F are Sylow 3-subgroups of G, they are conjugate to each other in G. Hence $(1_C)^G = (1_F)^G$. The group G has precisely seven irreducible characters $\chi_1, \dots, \chi_7$. Order them in such a way that $\chi_1 = 1_G$, $\chi_1(1) = \chi_2(1) = \chi_3(1) = 1$, $\chi_4(1) = 3$, $\chi_5(1) = \chi_6(1) = \chi_7(1) = 2$. Precisely one of the χ_5, χ_6, χ_7 is a rational valued character; let it be χ_5.

Now $(1_{C \cap F})^G = (1_{\{1\}})^G = \sum_{i=1}^{7} \chi_i(1)\chi_i$. Observe that

$$(1_C)^G - 1_G = \chi_4 + \chi_6 + \chi_7.$$

(As to the last result, compare it with the more general outcome of Satz V.17.13 of [3], in which the character theory of a class of Frobenius-like groups is described: note that $SL(2,3)$ is a member of that class).

Hence

$$\psi := (1_{C \cap F})^G + 1_G - (1_C)^G - (1_F)^G = \chi_2 + \chi_3 + \chi_4 + 2\chi_5.$$

Assume $\psi = \sum_{i=1}^{t} a_i \mu_i$, $a_i \in \mathbb{Q}_{\geq 0}$, where each μ_i a monomial character of G. Then at least one of the μ_i should have χ_5 as an irreducible constituent, but not any of χ_6 and χ_7 as irreducible constituent. Note further that 1_G is not an irreducible constituent of any of the μ_i.

Here is the list of all the monomial characters γ of G with $(\gamma, 1_G) = 0$ and $(\gamma, \chi_5) \geq 1$: $2(\chi_5 + \chi_6 + \chi_7)$; $\chi_5 + \chi_6 + \chi_7$; $\chi_5 + \chi_6$; $\chi_5 + \chi_7$; $\tilde{\gamma}$, with $\tilde{\gamma}$ the well-defined monomial character of G with $\tilde{\gamma}(1) = 8$, $(\tilde{\gamma}, \chi_6) = 1$, $(\tilde{\gamma}, \chi_7) = 0$; $\hat{\gamma}$, with $\hat{\gamma}$ the well-defined monomial character of G with $\hat{\gamma}(1) = 8$, $(\hat{\gamma}, \chi_7) = 1$, $(\hat{\gamma}, \chi_6) = 0$.

This shows that ψ is *not* of the form $\sum_{i=1}^{t} a_i \mu_i$, $a_i \geq 0$, $a_i \in \mathbb{Q}$, $t \geq 1$, each μ_i a monomial character of G!

By the way, it is quite remarkable that for any $L < SL(2,3)$ with $SL(2,3) = \langle K, L \rangle$, $K \neq L$, $K < SL(2,3)$, $|K| \neq 3$, it does hold that

$$(1_{K \cap L})^{SL(2,3)} + 1_{SL(2,3)} - (1_K)^{SL(2,3)} - (1_L)^{SL(2,3)}$$

is a sum of monomial characters of $SL(2,3)$. This can be verified by inspection as the second author for this paper did. □

REMARK. It is a result due to R.P. Langlands, that $L(s, \chi, \Omega/K)$ is indeed an entire function whenever $\Omega \supseteq K$ is a galois extension such that $\mathrm{Gal}(\Omega/K) \cong SL(2,3)$, for any 2-dimensional irreducible character χ of $\mathrm{Gal}(\Omega/K)$. Therefore, as the other irreducible characters of $\mathrm{Gal}(\Omega/K)$ are monomial, it holds that Artin's conjecture is true for $SL(2,3)$, i.e. each $L(s, \eta, \Omega/K)$, with η a character of $\mathrm{Gal}(\Omega/K) \cong SL(2,3)$ and $(\eta, 1_{\mathrm{Gal}(\Omega/K)}) = 0$, is an entire function. The same kind of argument applies on the solvable group $GL(2,3)$. Namely, it is a consequence to work of J. Tunnell that, given Δ/F galois with $\mathrm{Gal}(\Delta/F) \cong GL(2,3)$, it is a fact that each $L(s, \varphi, \Delta/F)$, where φ is a character of $\mathrm{Gal}(\Delta/F)$ with $(\varphi, 1_{\mathrm{Gal}(\Delta/F)}) = 0$, is an entire function. Thus in both the situations of $SL(2,3)$ and $GL(2,3)$, Brauer's problem has an affirmative answer!

We just saw in Example 7 that such a result for $SL(2,3)$ cannot be obtained merely by character-theoretical methods when dealing with $SL(2,3)$ alone. It will be shown in Example 9 that such a phenomenon happens also to be the case for the group $GL(2,3)$.

For the convenience of the reader we have given a suitable selection of references in which the theme we dealt with, is treated; see the references [10.a]-[10.i].

Let us pass to the group A_5 (the alternating group on five symbols).

EXAMPLE 8. Let $G=A_5$. There are certainly two subgroups C and F, each of order 5, such that $G=\langle C,F\rangle$; for instance, take $C=\langle(12345)\rangle$ and $F=\langle(21345)\rangle$. The group G has precisely five irreducible characters, namely $\chi_1,\dots,\chi_5$ with $\chi_1=1_G$, $\chi_2(1)=3$, $\chi_3(1)=3$, $\chi_4(1)=4$, $\chi_5(1)=5$. As C and F are conjugate to each other in A_5, we have that $(1_C)^G=(1_F)^G$. Now, as $(1_C)^G=1_G+\chi_2+\chi_3+\chi_5$, it holds that

$$\alpha:=(1_{C\cap F})^G+1_G-(1_C)^G-(1_F)^G=\chi_2+\chi_3+4\chi_4+3\chi_5.$$

Any monomial character μ of G not containing 1_G as irreducible constituent, happens to be a linear combination of five monomial characters $\vartheta_1,\dots,\vartheta_5$ with $\mu=\sum_{i=1}^5 n_i\vartheta_i$ with $n_i\in\mathbb{Z}_{\geq 0}$; here $\{\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5\}=\{\chi_5,\chi_2+\chi_3,\chi_2+\chi_3+\chi_4,\chi_2+\chi_4+\chi_5,\chi_3+\chi_4+\chi_5\}$. Assume now that $\alpha=\sum_{i=1}^5 a_i\vartheta_i$, where $a_i\geq 0$, $a_i\in\mathbb{Q}$, $i=1,\dots,5$. Then we can rewrite this for some positive integer t, as $t\alpha=\sum_{i=1}^5 b_i\vartheta_i$, $b_i\in\mathbb{Z}_{\geq 0}$. However, as the irreducible characters of G are linearly independent over $\mathbb{Q}$, we conclude that (omitting the calculation here) $t=0$, by solving a system of equations that arise. Hence α is not of the required form. Thus once again Brauer's problem might have an anwer in the affirmative for all cases for the group A_5 by inventing a method of reasoning lying beyond the scope of the theory of the monomial characters. □

EXAMPLE 9. Let G be the solvable group $GL(2,3)$ consisting of all the invertible 2×2-matrices over the field of three elements. It holds that $G=\langle S_1,S_2\rangle$, where $S_1=\langle\left(\begin{smallmatrix}-1&1\\-1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}-1&0\\-1&1\end{smallmatrix}\right)\rangle$ and $S_2=\langle\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1&0\\-1&1\end{smallmatrix}\right)\rangle$. We see that $S_1\cap S_2=\langle\left(\begin{smallmatrix}-1&0\\-1&1\end{smallmatrix}\right)\rangle$, a non-central subgroup of $GL(2,3)$ of order 2. The group G admits eight irreducible characters $\psi_1=1_G$, $\psi_2,\dots,\psi_8$ whose indexing we have adopted from [8]. We have $\psi_2(1)=1$, $\psi_3(1)=\psi_4(1)=\psi_5(1)=2$, $\psi_6(1)=\psi_7(1)=3$, $\psi_8(1)=4$. Precisely one of the characters ψ_3,ψ_4,ψ_5 has all its character values in $\mathbb{Z}$; let it be ψ_5. Note that the characters ψ_3 and ψ_4 are each others complex conjugate.

A direct calculation shows that

$$(1_{S_1\cap S_2})^G=\psi_1+\psi_3+\psi_4+\psi_5+2\psi_6+\psi_7+2\psi_8,$$

and that

$$(1_{S_1})^G=\psi_1+\psi_6+\psi_8=(1_{S_2})^G.$$

Therefore we find that

$$(1_{S_1 \cap S_2})^G + 1_G - (1_{S_1})^G - (1_{S_2})^G = \psi_3 + \psi_4 + \psi_5 + \psi_7.$$

Put $\chi = \psi_3 + \psi_4 + \psi_5 + \psi_7$. We argue that there are no $a_i \in \mathbb{Q}$, $a_i \geq 0$, and monomial characters ϑ_i of G, such that $\chi = \sum_{i=1}^{<\infty} a_i \vartheta_i$. Thus what one has to do is a search for monomial characters ϑ_i of G in which none of the 1_G, ψ_2, ψ_6 and ψ_8 occurs as an irreducible constituent. Although such monomial characters exist (ψ_5 and ψ_7 among others) it follows from the investigations of the second author of this paper that χ cannot be built in the way we want. We omit the vast amount of details in the verification of this fact. Also, as the reader can observe by inspection, the just sketched situation is the only kind of possibilities in which a non-trivial "Brauer configuration" $(1_{H \cap T})^{GL(2,3)} + 1_{GL(2,3)} - (1_H)^{GL(2,3)} - (1_T)^{GL(2,3)}$, $GL(2,3) = \langle H, T \rangle$, is not a sum of monomial characters of $GL(2,3)$. Such a phenomenon happened also to be the case, as we saw, for the groups $SL(2,3)$ and A_5. So, in a loose sense, there are only a few misses for an answer in the affirmative for Brauer's problem by means of character theory of groups alone, for the groups we have considered. □

We end with an exercise.

EXERCISE. Let G be a group such that $G = NT$, where $\{1\} \neq T < G$ and $\{1\} \neq N \ntrianglerighteq G$. Assume there exists a non-trivial abelian normal subgroup A of G such that $G = TA$. Then $(1_{N \cap T})^G + 1_G - (1_N)^G - (1_T)^G$ is a sum of monomial characters of G. (Hint: Brauer–Aramata, van der Waall–Uchida).

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